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Higher Order Generation of Exactly Solvable Supersymmetric Systems

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Starting from an exactly solvable potential, it is shown that it is always possible to construct an infinite set of higher order generation supersymmetric systems which are also exactly solvable. A method of deriving these potentials is presented and the corresponding eigenfunctions and eigenspectra are discussed. The theory is illustrated with an example in which the first- and second-order generation systems are considered.

1. INTRODUCTION

In a previous paper [1] it was established that if a potential $V(x)$ defined in a certain domain D is exactly solvable, that is, if the whole set of eigenfunctions and eigenvalues of the Schrödinger equation can be obtained by algebraic means, then it is always possible, relative to any state $|n\rangle$, to construct another $V^{(n)}$ defined in a domain D_n which also is exactly solvable.

In that work it was also hinted that extension to higher order generation potentials $V^{(n,m,\ldots)}$ involving more than one state is possible. The purpose of the present paper is to complete these hints by examining the problem from a more general point of view.

To facilitate the transition from earlier works we begin with a brief review of some essential features of the technique used in ref. 1 which will be needed below, adopting the same notations. However, the growing degree of complexity in the ensuing discussion requires better and more concise forms of notations; the improvements will gradually be added and explained as necessary.

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2. FORMULATION

Consider the matrix differential equation

$$\phi' + F\phi = 0; \qquad \phi = (\phi_1, \phi_2)^+; \qquad F = \begin{pmatrix} u_1 & d_1 \\ 0 & u_2 \end{pmatrix} \tag{1}$$

u_1, u_2, d_1 in principle may be any analytic functions. The system (1) can be conveniently be dealt with (see ref. 1) by use of the "mixing function" defined by

$$\phi_{1,n} = -X_n\, \phi_2 \tag{2}$$

It can be shown that if this function is solution of the second-order differential equation

$$X_n'' - 2u_2 X_n' + [(u_2^2 - u_2') - (u_1^2 - u_1') - E_n] X_n = 0 \tag{3}$$

then the first component ϕ_1 of the system (1) is an eigenfunction of the Schrödinger equation corresponding to the excited state $|n\rangle$ and eigenvalue E_n:

$$\phi_{1,n}'' - [u_1^2 - u_1']\, \phi_{1,n} = E_n \phi_{1,n} \tag{4}$$

On the other hand, if $u_1 = u_2 = u$, then $u(x)$ can be considered as the superpotential in $SU(2)$ from which a couple of "partner potentials" can be inferred

$$V^{\mp} = u^2 \mp u'$$

Supersymmetry preserves the double degeneracy of the eigenspectra while the eigenfunctions $\phi_{1,n}$, $\bar{\phi}_{1,n}$ corresponding to V^-, V^+ up to a constant of normalization are given by

$$\phi_{1,n} \simeq -X_n e^{-\int u dx}; \qquad \bar{\phi}_{1,n} \simeq -X_n' e^{-\int u dx} \tag{5}$$

The ladder operator $\Lambda^{\mp} = d/d^2 \mp u(x)$ as well as the SUSY Hamiltonians $H_1 = A^- A^+$ and $H_2 = A^+ A^-$ must satisfy the following commutation and anticommutation rules:

$$\frac{1}{2}\{A^-, A^+\} = \frac{d^2}{dx^2} - u^2 \tag{6}$$

$$\frac{1}{2}[A^-, A^+] = u'$$

3. SINGLE-BASE SYSTEMS

From now on, and for simplicity, the first-order generation potential $V^{(n)}$ which is constructed from the state (base) $|n\rangle$ will be referred to as the single-base potential. The mixing function corresponding to this potential will be denoted by $X^{(n)}$ (in place of the former X_n); the reason for this choice will be appreciated below when the discussion extends to the case of multibase systems. For the single-base case, the superpotential is defined as

$$v^{(n)} = u - \frac{X^{(n)'}}{X^{(n)}}$$

and the matrix differential equations become

$$\phi^{(n)'} + F^{(n)}\phi^{(n)} = 0; \qquad \phi^{(n)} = (\phi_1^{(n)}, \phi_2^{(n)})^+; \qquad F^{(n)} = \begin{pmatrix} v^{(n)} & d_1^{(n)} \\ 0 & v^{(n)} \end{pmatrix} \quad (1')$$

Using the same procedure, if the eigenfunction $\phi_{1,m}^{(n)}$ corresponds to the $|m\rangle$ state in this base,

$$\phi_{1,m}^{(n)} = -X_m^{(n)}\phi_2^{(n)}$$

Exact solvability of the potential $V^{(n)-}$ means that the function $X_m^{(m)}$ must be a solution of the second-order differential equation

$$X_m^{(n)''} - 2v^{(n)}X_m^{(n)'} - A_m^{(n)}X_m^{(n)} = 0 \tag{8}$$

which is exactly solvable. The solutions are

$$X_m^{(n)} = \frac{X^{(m)}}{X^{(n)}}, \qquad A_m^{(n)} = E_m - E_n \qquad (m \geq n) \tag{9}$$

The Schrödinger equation corresponding to the first component is then

$$\begin{aligned} \phi_{1,m}^{(n)''} - V^{(n)-}\phi_{2,m}^{(n)} &= A_m^{(n)}\phi^{(n)} \\ \phi_{1,m}^{(n)} &\simeq -X_m^{(n)}e^{-\int u dx} \end{aligned} \tag{10}$$

The analytical form of the second component $V^{(n)+}$ of the couple $V^{(n)\pm}$ is given by

$$V^{(n)+} = u^2 + u' + 2\frac{X^{(n)'}}{X^{(n)}}\left[\frac{X^{(n)}}{X^{(n)}} - 2u\right] - E_n \tag{11}$$

and is defined in the domain $D^{(n)}$. The corresponding mixing function $\overline{X}_m^{(n)}$ is a solution of the equation

$$\overline{X}_m^{(n)''} - 2v^{(n)}\overline{X}_m^{(n)'} - [2v^{(n)'} + A_m^{(n)}]\overline{X}_m^{(n)} = 0 \tag{12}$$

which also is exactly solvable:

$$\overline{X}_m^{(n)} = X_m^{(n)'} \tag{13}$$

$X_m^{(n)}$ is given in (9). Therefore the eigenfunction of $V^{(n)+}$ is

$$\overline{\phi}_{1,m}^{(n)} \simeq \left(\frac{X^{(m)}}{X^{(n)}}\right)' X^{(n)}\, e^{-\int u dx} \qquad (m > n) \tag{14}$$

These are essentially the main results presented in ref. 1 and may be completed with the following remarks.

(a) The ladder operators from which the SUSY Hamiltonians $H_1^{(n)} = \Lambda^{(n)-}A^{(n)+}$, $H_2^{(n)} = A^{(n)+}A^{(n)-}$ are constructed must satisfy the commutation and anticommutation rules

$$A^{(n)\mp} = \frac{d}{dx} \mp v^{(n)}; \qquad \frac{1}{2}\{A^{(n)-}, A^{(n)+}\} = \frac{d^2}{dx^2} - \left[u^2 + \frac{X^{(n)'}}{X^{(n)}}\left(\frac{X^{(n)'}}{X^{(n)}} - 2u\right)\right]$$

$$\frac{1}{2}[A^{(n)-}, A^{(n)+}] = u' - \frac{X^{(n)'}}{X^{(n)}}\left(\frac{X^{(n)'}}{X^{(n)}} - 2u\right) - E_n \tag{6$'$}$$

which can be compared with (6).

(b) Concerning the first component $V^{(n)-}$, note that for any chosen state (base) $|n>$, we always have

$$V^{(n)-} = V^- + \text{const} \tag{15}$$

which means that this component does not bring anything new to the present problem.

(c) The second component $V^{(n)+}$ given by (11) is more interesting and may lead to the construction of new exactly solvable potentials (see also ref. 3).

(d) From the theoretical point of view, the special case $m = n$ is interesting since it can be interpreted as representing the "ground state" of the first component $V^{(n)-}$ with eigenvalue $A_n^n = 0$. However, this "ground state" cannot be defined for the second component $V^{(n)+}$ [see relation (13)], so that the eigenspectra of the two components are not strictly degenerate. In other words, this remark means that, regardless of the nature of the symmetry (nonbroken or broken symmetry) of the "parent" couple $V^{\mp}$, the present construction always leads to systems with broken symmetry.

4. ISOSPECTRALITY

In order to go a step further in dealing with a multibase-system theory, it would be convenient to proceed through an intermediate stage concerning the isospectrality of the former single-base system. In fact, relative to each couple $V^{(n)\mp}$, it is possible to construct a family of potentials $V^{(n,\mu)\mp}$ (μ is an arbitrary constant) isospectral to it, that is, potentials which may sustain the same eigenspectra.

Isospectrality can be approached with the C transformation technique discussed in ref. 4.

According to this technique, the superpotential is defined by

$$v^{(n,\mu)} = v^{(n)} \mp \frac{c'}{c}$$

where $c(x, \mu)$ is in principle an unknown function. Isospectrality imposes the constraint

$$v^{(n,\mu)2} \mp v^{(n,\mu)'} = v^{(n)2} \mp v^{(n)'}$$

Taking the minus sign, for instance, it can be seen that this function $c(x, \mu)$ must be a solution of the equation

$$c'' - 2v^{(n,\mu)}c' = 0 \tag{16}$$

In solving (16) and noting that the second component $\phi_2^{(n,\mu)}$ in the system similar to (1) can be written as $\phi_2^{(n,\mu)} \simeq \exp[-\int v^{(n,\mu)}\, dx|$, we have

$$C(x) = \int|\phi_2^{(n,\mu)}|^{-2}dx + \mu \tag{17}$$

which is defined in a domain $D(n,\mu)$.

Therefore the couple $V^{(n,\mu)\mp}$ must depend on the choice of the constant of integration μ. Their exact solvability can be shown by using the same reasoning as above. In particular, the two mixing functions $X^{(n,\mu)}$ and $\overline{X}^{(n,\mu)}$ corresponding to the couple must be a solution of the equations

$$X_m^{(n,\mu)''} - 2v^{(n,\mu)}X_m^{(n,\mu)'} - A_m^{(n)}X_m^{(n,\mu)} = 0 \tag{18}$$

$$\overline{X}_m^{(n,\mu)''} - 2v^{(n,\mu)}\overline{X}_m^{(n,\mu)'} - |2v^{(n,\mu)'} + A_m^{(n)}|\overline{X}_m^{(n,\mu)} = 0 \tag{19}$$

The solution of the first one is

$$X_m^{(n,\mu)} = \frac{X_m^{(n)}}{C(x)}$$

$c(x)$ given by (17), and the eigenfunction pertaining to $V^{(n,\mu)-}$ is simply

$$\phi_m^{(n,\mu)} \simeq -X_m^{(n,\mu)}\phi_2^{(n,\mu)} \tag{20}$$

The solution of the second one is simply $\overline{X}^{(n,\mu)} = X^{(n,\mu)'}$ and the eigenfunction corresponding to $V^{(n,\mu)+}$ can be written as

$$\overline{\phi}_m^{(n,\mu)} \simeq -\overline{X}_m^{(n,\mu)}\phi_2^{(n,\mu)} \tag{21}$$

Remarks. (a) It can be verified that

$$V^{(n,\mu)-} = V^- + E_n$$

which means independence in the choice of the parameter μ for the first component of the couple

(b) The analytical form of the second component $V^{(n,\mu)+}$ is

$$V^{(n,\mu)+} = u^2 + u' + 2\frac{X^{(n)'}}{X^{(n)}}\left[\frac{X^{(n)'}}{X^{(n)}} - 2u\right] + 2\frac{C(x,\mu)}{C(x,\mu)}\left[\frac{C(x,\mu)}{C(x,\mu)} + 2\frac{X^{(n)'}}{X^{(n)}} - 2u\right] \tag{22}$$

which obviously depends on the choice of μ. Its eigenfunction is, after (21),

$$\overline{\phi}_m^{(n,\mu)} \simeq -\overline{X}_m^{(n,\mu)} X^{(n)} C(x,\mu) e^{-\int u dx} \tag{23}$$

(c) In the case where isospectrality is not required, the function $c(x)$ will not be subject to the constraint (16), so that other choices are equally possible. For example:

(c1) If $c(x) = X^{(n)}$, with $X^{(n)}$ being the mixing function given in (8), then this approach reduces to the case of the single-base system discussed above.

(c2) On the other hand, if $c(x) = \phi_n$, with ϕ_n being one of the eigenfunctions of V^-, then this method will lead ultimately to the formulation found in ref. 3.

This explains why the two cases (c1) and (c2) must yield equivalent results.

5. THE DOUBLE-BASE SYSTEMS

Now we consider what happens when one wishes to include more than one base in the formulation.

According to the preceding discussion, this will obviously depend on the right choice of the function $c(x)$. We consider two significant cases, case A and case B (although from a more general point of view, it can be expected that this number will not necessarily be limited to two; this aspect is under investigation).

Case A. Recall that for the case of a single base, the mixing function was denoted by $X_m^{(n)}$ implying the following meaning: the upper index (n) refers to the chosen base $|n\rangle$, while the second one m indicates the order of the excited state $|m\rangle$ ($s > m > n$, etc.).

In the case of a double-base system it will therefore be natural to denote the mixing function as $X_s^{(n,m)}$ in which (n, m) correspond respectively to the first and second bases $|n\rangle$ and $|m\rangle$ and s to the excited state $(s > m > n)$.

The superpotential in Case A involves two quantities: $[c(x) = X^{(n,m)}]$

$$A_v(n, m) = v^{(n)} - \frac{X^{(n,m)\prime}}{X^{(n,m)}} \tag{24}$$

$v^{(n)}$ corresponds to the superpotential of the preceding (first) system, while the second term involves the mixing function of the double base $X^{(n,m)}$. From (9) we already know that $X^{(n,m)} = X^{(m)}/X^{(n)}$, which can always be determined since the "parent" potential is assumed to be exactly solvable.

Following the same procedure, it can be verified that this mixing function $X_s^{(n,m)}$ must be a solution of the second-order differential equation

$$X_s^{(n,m)\prime\prime} - 2^{\mathrm{A}}\, v^{(n,m)} X_s^{(n,m)\prime} - A_s^{(n,m)} X_s^{(n,m)} = 0 \tag{25}$$

which is exactly solvable with the solution

$$X_s^{(n,m)} = \frac{X^{(n,s)}}{X^{(n,m)}}; \qquad A_s^{(n,m)} = E_s - E_m \tag{26}$$

From the superpotential ${}^{\mathrm{A}}v^{(n,m)}$, the two partner potentials ${}^{\mathrm{A}}V^{(n,m)\mp}$ can be inferred,

$${}^{\mathrm{A}}V^{(n,m)\mp} = {}^{\mathrm{A}}v^{(n,m)2} \mp {}^{\mathrm{A}}v^{(n,m)\prime}$$

Making use of (25), we have

$${}^{\mathrm{A}}V^{(n,m)-} = u^2 - u' + E_m$$

$${}^{\mathrm{A}}V^{(n,m)+} = u^2 + u' + 2\left[\frac{X^{(n)\prime}}{X^{(n)}} + \frac{X^{(n,m)\prime}}{X^{(n,m)}}\right]\left[\frac{X^{(n)\prime}}{X^{(n)}} + \frac{X^{(n,m)\prime}}{X^{(n,m)}} - 2u\right] - E_n \tag{27}$$

Discussion. (a) For the first member ${}^{\mathrm{A}}V^{(n,m)-}$ of the couple we always have the property

$${}^{\mathrm{A}}V^{(n,m)-} = V^- + E_m \tag{28}$$

E_m is a constant depending on the choice of the bases.

(b) For the second member ${}^{\mathrm{A}}V^{(n,m)+}$, by definition, we have

$$\frac{X^{(n,m)\prime}}{X^{(n,m)}} = \frac{X^{(m)\prime}}{X^{(m)}} - \frac{X^{(n)\prime}}{X^{(n)}}$$

so that (27) leads to the following second property:

$${}^{\mathrm{A}}V^{(n,m)+} = V^{(m)+}, \qquad m > n \tag{29}$$

As $|n\rangle$ and $|m\rangle$ are chosen arbitrarily, Case A in fact does not bring anything new in the theory. Note also that the relation (27) has already been discussed in ref. 1, but with the use of a slightly different type of notation. However, it may become useful in quasi-exactly solvable problems, which will be discussed later in a more appropriate context.

Case B. The choice for this case is $c(x) = \overline{X}^{(n,m)}$, with $\overline{X}^{(n,m)}$ given by (12) $[\overline{X}^{(n,m)} = X^{(n,m)'}$, $X^{(n,m)}$ is a solution of equation (8)].

The construction of the corresponding superpotential ${}^{\mathrm{B}}v^{(n,m)}$ follows as in Case A, that is,

$$^{\mathrm{B}}v^{(n,m)} = v^{(n)} - \frac{\overline{X}^{(n,m)'}}{\overline{X}^{(n,m)}}$$

from which the two partner potentials ${}^{\mathrm{B}}V^{(n,m)\mp}$ can be inferred,

$$^{\mathrm{B}}V^{(n,m)\mp} = {}^{\mathrm{B}}v^{(n,m)2} \mp {}^{\mathrm{B}}v^{(n,m)'}$$

Exact solvability of these potentials can be proven by returning to the original system of matrix equation similar to (1); for instance, with the first member

$$^{\mathrm{B}}\phi^{(n,m)'} + {}^{\mathrm{B}}F^{(n,m)}\, {}^{\mathrm{B}}\phi^{(n,m)} = 0;$$

$$^{\mathrm{B}}\phi^{(n,m)} = ({}^{\mathrm{B}}\phi_1^{(n,m)}, {}^{\mathrm{B}}\phi_2^{(n,m)})^{+}; \qquad {}^{\mathrm{B}}F^{(n,m)} = \begin{pmatrix} {}^{\mathrm{B}}V^{(n,m)} & {}^{\mathrm{B}}\phi_1^{(n,m)} \\ 0 & {}^{\mathrm{B}}V^{(n,m)} \end{pmatrix}$$

with the corresponding mixing function ${}^{\mathrm{B}}X^{(n,m)}$ defined by (the index B is included to avoid any confusion with Case A)

$$^{\mathrm{B}}\phi_{1,s}^{(n,m)} \simeq -{}^{\mathrm{B}}X_s^{(n,m)}\, {}^{\mathrm{B}}\phi_2^{(n,m)} \tag{30}$$

This mixing function must be a solution of the equation

$$^{\mathrm{B}}X_s^{(n,m)''} - 2\,{}^{\mathrm{B}}v^{(n,m)}\, {}^{\mathrm{B}}X_s^{(n,m)} - A_s^{(n,m)}\, {}^{\mathrm{B}}X_s^{(n,m)} = 0 \tag{31}$$

which is exactly solvable.

The solutions are

$$^{\mathrm{B}}X_s^{(n,m)} = \frac{\overline{X}^{(n,s)}}{\overline{X}^{(n,m)}}, \qquad A_s^{(n,m)} = E_s - E_m \tag{32}$$

Its eigenfunction can be written after simplifications as

$$^{\mathrm{B}}\phi_{1,s}^{(n,m)} \simeq -\left(\frac{X^{(s)}}{X^{(n)}}\right)' X^{(n)} e^{-\int u dx} \qquad (s > m > n) \tag{33}$$

Note the following property:

$$^{B}V^{(n,m)-} = V^{(n)+} + \text{Const} \tag{34}$$

which, in some sense, may express the "shape invariance" character of the couple $^{B}V^{(n,m)-}$, $V^{(n)+}$ (it must not be confused with the concept of shape invariance in the sense of Gedenshtein [5]).

Concerning the second member $^{B}V^{(n,m)+}$, this matrix equation becomes

$$^{B}\overline{\phi}^{(n,m)'} + {}^{B}\overline{F}^{(n,m)}\,{}^{B}\overline{\phi}^{(n,m)} = 0;$$

$$^{B}\overline{\phi}^{(n,m)} = ({}^{B}\overline{\phi}_1^{(n,m)}, {}^{B}\phi_2^{(n,m)})^{+}; \qquad {}^{B}\overline{F}^{(n,m)} = \begin{pmatrix} -{}^{B}v^{(n,m)} & {}^{B}\overline{d}_1^{(n,m)} \\ 0 & {}^{B}V^{(n,m)} \end{pmatrix}$$

with the mixing function $^{B}\overline{X}^{(n,m)}$

$$^{B}\overline{\phi}_{1,s}^{(n,m)} = -{}^{B}\overline{X}_s^{(n,m)}\,{}^{B}\phi_2^{(n,m)}$$

which must be a solution of the following equation:

$$^{B}\overline{X}_s^{(n,m)} - 2\,{}^{B}v^{(n,m)}\,{}^{B}\overline{X}_s^{(n,m)'} - [2\,{}^{B}v^{(n,m)'} + A_s^{(n,m)}]\,{}^{B}\overline{X}_s^{(n,m)} = 0 \tag{35}$$

Obviously,

$$^{B}\overline{X}_s^{(n,m)} = {}^{B}X_s^{(n,m)'}$$

so that the analytical form of $^{B}V^{(n,m)+}$ is

$$^{B}V^{(n,m)+} = v^{(n)2} - v^{(n)'} + 2\frac{^{B}\overline{X}^{(n,m)'}}{^{B}X^{(n,m)}}\left[\frac{^{B}\overline{X}^{(n,m)'}}{^{B}X^{(n,m)}} + 2\frac{X^{(n)'}}{X^{(n)}} - 2u\right] - A^{(n,m)} \tag{36}$$

After simplification and using (35), the analytical form of the eigenfunctions can be found:

$$^{B}\phi_{1,s}^{(n,m)} \simeq -\left(\frac{(x^{(s)}/x^{(n)})'}{(x^{(m)}/x^{(n)})'}\right)'\left(\frac{x^{(m)}}{x^{(n)}}\right)' x^{(n)} e^{-\int u dx} \tag{37}$$

An example is given in the Appendix that tests the above results.

The method can be extended to the more general case of Nth-order generation (or N multiple bases), but at the cost of a rapidly growing complexity. For instance, with the third-order generation one has superpotentials of type $V^{(n,m,s)}$. Since in the preceding generation there are two superpotentials $^{A}V^{(n,m)}$ and $^{B}V^{(n,m)}$ we have four possibilities to construct $V^{(n,m,s)\pm}$, which in turn lead to four couple of exactly solvable potentials.

More generally, corresponding to the Nth generation, there will be 2^N or 2^{N+1} such potentials. This number is reduced, however, with constraints of type (15), (28), or (34).

The main results presented in this work are summarized as follows:

- For the single-base system (first-order generation) the method confirms the point of view discussed in our previous paper [1] as well as work by other authors.
- Coupled with the "C transformation" technique, this method enables one to construct the family of potentials isospectral to the one corresponding to this single-base system [see (22)].
- Extension to multibase systems is possible, leading to the construction of new types of exactly solvable potentials [see (37)].

APPENDIX

In order to illustrate the theory, it is appropriate to begin with an example simple enough from the mathematical point of view that the interested reader can directly verify the mechanism of construction of the results and test the consistency of the method. The other, more elaborate examples have also been worked out and will be presented in subsequent papers.

The square-well potential with infinite wall, which usually serves to simulate a model of confinement in physics, will be defined as

$$V(x) = 0, \qquad 0 < x < \pi, \qquad D: \quad (0, \pi)$$

$$V(x) = \infty, \qquad x < 0; \qquad x > \pi$$

The system (1) corresponds to the case $u_1 = u_2 = 0$. The solutions are

$$\phi_{1,n} \simeq \sin nx, \qquad |E_n| = m^2, \qquad n = 1, 2, 3, \ldots$$

A1. Single-Base System

Here we test the results (11), (14), and (15). Assume $n = 1$, so that

$$V^{(1)-} = -1, \qquad V^{(1)+} = -1 + \frac{2}{\sin_x^2}$$

The eigenfunctions and eigenvalues are as follows:

m	$\phi_{1,m}^{(1)}$	$\lvert E_m^{(1)}\rvert$
2	$2\sin^2 x$	3
3	$8\sin^2 x\cos x$	8
4	$4\sin^2 x[-1 + 6\cos^2 x]$	15

For $n = 2$

$$V^{(2)-} = -4, \qquad V^{(2)+} = -4 + \frac{8}{\sin^2 2x}$$

and we have the following results:

3	$\sin^2 x \left[5\cos x + \dfrac{\sin^2 x}{\cos x} \right]$	5
4	$4\sin^2 2x$	12
5	$5\cos x - \dfrac{2\sin 5x \cos 2x}{\sin 2x}$	21

Remarks. (a) Regardless of the choice of the base $|n\rangle$, the first components $V^{(n)-}$ are similar and represent the original square well, but what is changed is the "bottom" of this well.

(b) However, the second component behaves quite differently according to the choice of the base and becomes a non-square-well potential. For instance, its domain is $(0, \pi)$ for $n = 1$, but for $n = 2$ it becomes a double well separated by an infinite wall at $\pi/2$.

A2. Single-Base Isospectrality

Here we test the results (21) and (22).

As the constant of integration μ is arbitrary, the simplest choice would be $\mu = 0$;

$$\frac{c(x,0)'}{c(x,0)} = -\frac{2}{\sin 2x}$$

Therefore, for $n = 1$,

$$V^{(1,0)+} = -1 + \frac{2}{\cos^2 x}$$

and we have the following results:

2	$2\cos^2 x$	3
3	$3\cos 3x - \tan x \sin 3x$	8

A3. Double-Base Systems

Here we test the results (36) and (37) with the conditions $n = 1$ and $m = 2$. Thus

$$ {}^{B}V^{(1,2)-} = -4 + \frac{2}{\sin^2 x}; \qquad {}^{B}V^{(1,2)+} = -4 + \frac{6}{\sin^2 x} $$

and we have:

s	${}^{B}\phi_{1,s}^{(1,2)}$	$\lvert E_s^{(1,2)}\rvert$
3	$8 \sin^3 x$	5
4	$48 \sin^3 x \cos x$	12
5	$24 \sin^3 x[7 \cos^2 x - \sin^2 x]$	21

Remarks. (a) Note that ${}^{B}V^{(1,2)-} = V^{(1)+} - 3$, which confirms (34).
(b) For the present special case, consider the case $n = 1$, $m = 3$, so that

$$ {}^{B}V^{(1,3)+} = -16 + \frac{8}{\sin^2 2x} + \frac{4}{\sin^2 x} = V^{(1)+} + V^{(2)+} - 11 $$

This result constitutes a still scarce but nontrivial example in quantum mechanics where the combination of two exactly solvable potentials ($V^{(1)+}$ and $V^{(2)+}$) may lead to another potential (${}^{B}V^{(1,3)+}$ which itself is also exactly solvable.

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